JOURNAL OF APPROXIMATION THEORY 46, 137–156 (1986)

Local and Global Lipschitz Constants

JAMES R. ANGELOS, MYRON S. HENRY,
EDWIN H. KAUFMAN, JR. AND TERRY D. LENKER

Department of Mathematics, Central Michigan University, Mount Pleasant, Michigan 48859, U.S.A.

AND

ANDRÁS KROÓ*

Mathematical Institute of the Hungarian Academy of Sciences, Budapest, Hungary

Communicated by Oved Shisha

Received September 26, 1984

DEDICATED TO THE MEMORY OF GÉZA FREUD†

Let X be a closed subset of $I=[-1, 1]$, and let $B_n(f)$ be the best uniform approximation to $f \in C[X]$ from the set of polynomials of degree at most n. An extended global Lipschitz constant is defined for f, and it is shown that this constant is asymptotically equivalent to the strong unicity constant. Estimates of the size of the local Lipschitz constant for f are given when the cardinality of the set of extremal points of $f-B_n(f)$ is $n+2$. Examples which illustrate that the local and extended global Lipschitz constants may have very different asymptotic behavior are constructed.

1. INTRODUCTION

Let X be a closed subset of $I=[-1, 1]$ which contains at least $n+2$ points, and suppose $f \in C[X]$, the space of continuous, real-valued functions on X endowed with the uniform norm $\|\cdot\|$. Denote the set of all polynomials of degree n or less by Π_n. The behavior of the global Lipschitz and the strong unicity constants determined by f, X, and n has been the

* Research for this paper was done while this author was a visiting professor at Central Michigan University, August 1983–May 1984.

† Without G. Freud's pioneer work on Lipschitz constants, the results of this and many other papers would not have been possible.

0021-9045/86 $3.00

focus of a number of research papers during the last decade [2, 3, 8, 12, 16].

More recently, two papers [1, 9] have investigated the behavior of local Lipschitz and strong unicity constants. In the present paper we continue the investigation of the behavior of local and global Lipschitz constants, with an emphasis on how local and global Lipschitz constants relate to each other, and to strong unicity and Lebesgue constants.

Before stating some known results alluded to in the previous paragraphs, we establish notation that will be used throughout the paper. Let the best approximation to f from Π_n be designated by $B_n(f)$. The error function is then

$$e_n(f)(x) = f(x) - B_n(f)(x), \tag{1.1}$$

and the extremal set for the error function is

$$E_n(f) = \{x \in X \colon |e_n(f)(x)| = \|e_n(f)\|\}. \tag{1.2}$$

An alternant of the error function is any set

$$X_n = \{x_0, x_1, ..., x_{n+1}\} \subseteq E_n(f)$$

for which $e_n(f)(x_i) = \gamma(-1)^i \|e_n(f)\|$, $i = 0, 1, ..., n+1$, where $\gamma = \operatorname{sgn} e_n(f)(x_0)$.

DEFINITION 1. For $f \in C[X]$, the global Lipschitz constant is defined as

$$\lambda_n(f) = \sup_{\substack{g \neq f \\ g \in C[X]}} \frac{\|B_n(f) - B_n(g)\|}{\|f - g\|}. \tag{1.3}$$

DEFINITION 2. For $f \in C[X]$, the strong unicity constant is defined as

$$M_n(f) = \sup_{\substack{p \in \Pi_n \\ p \neq B_n(f)}} \frac{\|p - B_n(f)\|}{\|f - p\| - \|f - B_n(f)\|}. \tag{1.4}$$

DEFINITION 3. For $f \in C[X]$, let

$$\lambda_n(f, \delta) = \sup_{\substack{0 < \|f - g\| \leqslant \delta \\ g \in C[X]}} \frac{\|B_n(f) - B_n(g)\|}{\|f - g\|}. \tag{1.5}$$

Then

$$\hat{\lambda}_n(f) = \lim_{\delta \to 0^+} \lambda_n(f, \delta) \tag{1.6}$$

is the local Lipschitz constant.

The inequalities

$$\hat{\lambda}_n(f) \leqslant \lambda_n(f) \leqslant 2M_n(f) \tag{1.7}$$

follow easily from (1.3), (1.6), and [4, p. 82], and represent elementary relationships between these constants.

In the spirit of (1.5) and (1.6), it is also possible to define a local strong unicity constant [9] for each $f \in C[X]$. For rational approximation, it has been shown that local and global strong unicity constants may differ significantly. In the case of linear approximation, however, it is known that these constants are equal for each $f \in C[X]$ [9]. In contrast, we will find that for each $n > 0$ and for each $\varepsilon > 0$, there exist $f \in C[X]$ such that $\hat{\lambda}_n(f)/\lambda_n(f) < \varepsilon$.

The Lebesgue constant also plays a prominent role in the subsequent investigations.

DEFINITION 4. Let $\{l_i\}_{i=0}^{k}$ be the Lagrange basis functions for Π_k determined by $Y = \{y_0, y_1, ..., y_k\} \subseteq X$. The Lebesgue constant determined by Y is

$$\Lambda_k(Y) = \left\| \sum_{i=0}^{k} |l_i| \right\|. \tag{1.8}$$

The next definition is needed in Theorem 1 and appears in [14].

DEFINITION 5. Let $f, g \in C[X]$. Then

$$D_f B_n(g) = \lim_{t \to 0} \frac{B_n(f + tg) - B_n(f)}{t} \tag{1.9}$$

if the limit exists. In this case we say that $D_f B_n(g)$ is the derivative of $B_n(f)$ in the direction of g.

It was essentially shown in [14] that if the cardinality of $E_n(f)$, denoted by $|E_n(f)|$, equals $n+2$, then $D_f B_n(g)$ exists for all $g \in C[X]$ and $D_f B_n$ is a projection which maps $C[X]$ onto Π_n.

As we shall see in Theorem 1, certain interpolating polynomials provide characterizations of both $\hat{\lambda}_n(f)$ and $M_n(f)$. Specifically, for $X_n = \{x_0, x_1, ..., x_{n+1}\}$, define $\{q_i\}_{i=0}^{n+1} \subseteq \Pi_n$ by

$$q_i(x_j) = (-1)^j, \qquad j = 0, 1, ..., n+1, j \neq i, \tag{1.10}$$

and define $Q_{n+1} \in \Pi_{n+1}$ by

$$Q_{n+1}(x_j) = (-1)^j, \qquad j = 0, 1, ..., n+1. \tag{1.11}$$

THEOREM 1 [1, 8]. *Let $f \in C[X]$ and let $E_n(f)$ consist of precisely the $n+2$ points $x_0 < x_1 < \cdots < x_{n+1}$. Then*

$$\text{(a)} \quad \hat{\lambda}_n(f) = \left\| \sum_{i=0}^{n+1} \frac{|q_i|}{1 + |q_i(x_i)|} \right\| = \|D_f B_n\|, \tag{1.12}$$

and

$$\text{(b)} \quad M_n(f) = \max_{0 \leqslant i \leqslant n+1} \|q_i\|. \tag{1.13}$$

In Section 3 we will use an extension of (1.13) which does not require that $|E_n(f)| = n+2$. If $|E_n(f)| = n+2$, we observe from Theorem 1 that neither $\hat{\lambda}_n$ nor M_n actually depend on f; instead, they both depend only on the $n+2$ points of $E_n(f)$. Hence whenever $|X_n| = n+2$, we will employ the notation $\hat{\lambda}_n(X_n)$ and $M_n(X_n)$. The more common notation $\hat{\lambda}_n(f)$ and $M_n(f)$ will be used only if a given function f plays a central role in the analysis.

The next theorem gives upper and lower bounds for $\hat{\lambda}_n(X_n)$ in terms of $\Lambda_{n+1}(X_n)$.

THEOREM 2. *Let $X_n = \{x_0, ..., x_{n+1}\} \subseteq X$. Let $\hat{\lambda}_n(X_n)$ be the local Lipschitz constant given by* (1.12), *and let $\Lambda_{n+1}(X_n)$ be the Lebesgue constant determined by X_n. Then*

$$\Lambda_{n+1}(X_n) - \|Q_{n+1}\| \leqslant \hat{\lambda}_n(X_n) \leqslant \Lambda_{n+1}(X_n) + \|Q_{n+1}\| \leqslant 2\Lambda_{n+1}(X_n), \tag{1.14}$$

where Q_{n+1} is defined by (1.11).

Proof. Suppose X_n is the extremal set for $e_n(f)$, $f \in C[X]$. It has been shown [1] that

$$(D_f B_n)(g)(x) = B_n(g, X_n)(x), \qquad x \in X, \tag{1.15}$$

where $B_n(g, X_n)$ is the best approximation to g from Π_n on X_n. The result follows from (1.12), (1.15), and [(1.14), 5]. ∎

Example 1 demonstrates the utility of Theorem 2.

EXAMPLE 1. Let $f(x) = 1/(x-a)$, $x \in I$ and $|a| \geqslant 2$. If $X_n = E_n(f)$, then $|X_n| = n+2$. It can be shown [10] that the Q_{n+1} defined by (1.11) is given by

$$Q_{n+1}(x) = \frac{a}{n\sqrt{a^2-1}} (1-x^2)\, C_n'(x) - xC_n(x),$$

where $C_n(X)$ is the Chebyshev polynomial of degree n. Thus,

$$\|Q_{n+1}\| \leqslant 3\,|a|/\sqrt{a^2-1}. \tag{1.16}$$

It is also known [11] that there exist positive constants α and β not depending on n, such that $\alpha \log(n+1) \leqslant \Lambda_{n+1}(X_n) \leqslant \beta \log(n+1)$. In this case we say the precise order of $\Lambda_{n+1}(X_n)$ is $\log(n+1)$. This observation, (1.14), and (1.16) imply that $\hat{\lambda}_n(X_n)$ is of precise order $\log(n+1)$. As a point of interest, for this example $M_n(X_n)$ is of precise order n [10]. Therefore

$$\lim_{n\to\infty} [\hat{\lambda}_n(X_n)/M_n(X_n)] = 0.$$

The inequality (1.14) and Example 1 suggest the following question. For each sequence $\{X_n\}_{n=1}^{\infty}$, $X_n = \{x_0^{(n)}, x_1^{(n)}, ..., x_{n+1}^{(n)}\}$, does there exist a positive constant α not depending on n such that

$$\alpha \Lambda_{n+1}(X_n) \leqslant \hat{\lambda}_n(X_n)? \tag{1.17}$$

This question will be answered in the next section.

2. Lower Bounds

In this section we seek lower bounds for local Lipschitz constants. First we show that (1.17) is not always true.

Example 2. Let $X_n = \{x_i\}_{i=0}^{n+1} \subseteq I$, with $-1 = x_0 < x_1 < \cdots < x_{n+1} < 1$. If q_i, $i = 0, 1, ..., n+1$, is defined by (1.10), and l_i^j, $j = 0, ..., n+1$, $i \neq j$, are the Lagrange basis elements determined by $X_n - \{x_j\}$, then

$$|q_j(x)| \leqslant \sum_{\substack{i=0\\ i\neq j}}^{n+1} |l_j^i(x)|. \tag{2.1}$$

It is well known [15] that

$$l_i^j(x) = \frac{w_j(x)}{(x-x_i)\,w_j'(x_i)}, \qquad i \neq j, \tag{2.2}$$

where

$$w_j(x) = \prod_{\substack{i=0\\ i\neq j}}^{n+1} (x - x_i).$$

Expressions (2.1) and (2.2) imply that

$$|q_j(x)| \leqslant \sum_{\substack{i=0\\ i\neq j}}^{n+1} \left|\frac{w(x)(x_i - x_j)}{(x-x_i)(x-x_j)\, w'(x_i)}\right|,$$

where $w(x) = \prod_{i=0}^{n+1}(x - x_i)$.

Define $u_{ij}(x) = w(x)/(x-x_i)(x-x_j)$, for $i \neq j$, $i, j = 0, 1,..., n+1$, and let

$$U_n = \max_{\substack{i,j=0,...,n+1\\ i\neq j}} \|u_{ij}\|.$$

Then (1.13) implies that

$$M_n(X_n) \leqslant (x_{n+1} - x_0)\, U_n \sum_{i=0}^{n+1} \frac{1}{|w'(x_i)|}. \tag{2.3}$$

Suppose now that

$$x_{n+1} - x_0 = d_n \leqslant \tfrac{1}{4}. \tag{2.4}$$

If $x \in [-1, -\frac{1}{2}]$, then $|u_{ij}(x)| \leqslant (\frac{1}{2})^n$, for all i and j, $i \neq j$. Hence for some $i_0 \neq j_0$,

$$U_n = \|u_{i_0 j_0}\| = |u_{i_0 j_0}(1)| \leqslant \prod_{i=0}^{n+1} (1 - x_i) = \|w\|. \tag{2.5}$$

Combining (2.3), (2.4), and (2.5) yields

$$M_n(X_n) \leqslant d_n \sum_{i=0}^{n+1} \frac{\|w\|}{|w'(x_i)|}. \tag{2.6}$$

On the other hand,

$$2\Lambda_{n+1}(X_n) \geqslant \sum_{i=0}^{n+1} \frac{\|w\|}{|w'(x_i)|}. \tag{2.7}$$

Together (2.6) and (2.7) imply that

$$\frac{M_n(X_n)}{\Lambda_{n+1}(X_n)} \leqslant 2d_n. \tag{2.8}$$

Now (2.8) implies that

$$\lim_{d_n \to 0} \frac{M_n(X_n)}{\Lambda_{n+1}(X_n)} = 0. \tag{2.9}$$

If $\lim_{n\to\infty} d_n = 0$, we observe that we have constructed a sequence of point sets $\{X_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \frac{\hat{\lambda}_n(X_n)}{\Lambda_{n+1}(X_n)} = 0. \tag{2.10}$$

Furthermore, for an appropriate choice of the null sequence $\{d_n\}_{n=1}^{\infty}$, both (2.9) and (2.10) will converge to zero arbitrarily rapidly.

Henry *et al.* [12] have shown that

$$\Lambda_{n+1}(X_n) \leqslant \left(\frac{4}{d_n} + 1\right) M_n(X_n). \tag{2.11}$$

We thus observe that if there exists $\delta > 0$ such that $d_n \geqslant \delta$ for all n, then the behavior exhibited in (2.9) is not possible. For any choice of $d_n = x_{n+1} - x_0$, (2.11) implies that $d_n/6 \leqslant M_n(X_n)/\Lambda_{n+1}(X_n)$.

Referring again to (1.14), we may infer from (2.10) that for the point sets X_n constructed in Example 2, $\Lambda_{n+1}(X_n)$ and $\|Q_{n+1}\|$ have the same asymptotic order. Thus (1.14) does not always provide a useful lower bound for local Lipschitz constants.

Consequently, it is desirable to establish other lower bounds for $\hat{\lambda}_n$, perhaps still involving Lebesgue constants. To illustrate, the following theorem for strong unicity constants was essentially proven in [12].

THEOREM 3. *Let $X_n = \{x_0, x_1, \dots, x_{n+1}\} \subseteq X$ and let Λ^j_{n+1} be the Lebesgue constant determined by $X^j_n = X_n - \{x_j\}$, $j = 0, 1, \dots, n+1$. Then*

$$M_n(X_n) = \max_{0 \leqslant j \leqslant n+1} \Lambda^j_{n+1}. \tag{2.12}$$

The next theorem is a useful companion to Theorem 3 for the local Lipschitz constant. We use the notation $\|\cdot\|_{X_n}$ for the uniform norm restricted to X_n, and set $M^n_n(X_n) = \max_{0 \leqslant j \leqslant n+1} \|q_j\|_{X_n}$ for the polynomials q_j defined by (1.10).

THEOREM 4. *Let $X_n = \{x_0, x_1, \dots, x_{n+1}\} \subseteq X$ and let Λ^j_{n+1} denote the Lebesgue constant determined by $X^j_n = X_n - \{x_j\}$, $j = 0, \dots, n+1$. Then*

$$\min_{0 \leqslant j \leqslant n+1} \frac{\Lambda^j_{n+1}}{n+2} \leqslant \max_{0 \leqslant j \leqslant n+1} \frac{\Lambda^j_{n+1}}{1 + |q_j(x_j)|} \leqslant \hat{\lambda}_n(X_n)$$
$$\leqslant \frac{2M^n_n(X_n)}{1 + M^n_n(X_n)} \min_{0 \leqslant j \leqslant n+1} \Lambda^j_{n+1}. \tag{2.13}$$

Proof. It can be shown [1] that

$$B_n(g, X_n)(x) = \sum_{i=0}^{n+1} \frac{g(x_i)(-1)^{i+1} q_i(x)}{m_i}, \qquad x \in X, \tag{2.14}$$

where q_i, $i = 0, 1, \ldots, n+1$, is defined by (1.10); $m_i = 1 + |q_i(x_i)|$, $i = 0, 1, \ldots, n+1$; and $B_n(g, X_n)$ is defined below (1.15).

Since $B_n(g, X_n) \in \Pi_n$,

$$B_n(g, X_n)(x) = \sum_{\substack{i=0 \\ i \neq j}}^{n+1} h_i l_i^j(x), \qquad x \in X, \tag{2.15}$$

for an appropriate choice of the h_i, $i = 0, 1, \ldots, n+1$; $i \neq j$. Here $\{l_i^j\}_{i=0, i\neq j}^{n+1}$ are the Lagrange basis functions for Π_n determined by X_n^j. Evaluating (2.14) and (2.15) at x_k for $k = 0, 1, \ldots, n+1$ and $k \neq j$ yields

$$h_k = (-1)^{k+1} \sum_{\substack{i=0 \\ i \neq k}}^{n+1} \frac{(-1)^i g_i}{m_i} + g_k\left(1 - \frac{1}{m_k}\right), \tag{2.16}$$

where $g_i = g(x_i)$, $i = 0, 1, \ldots, n+1$.

To establish the upper bound in (2.13), suppose that $g \in \{u \in C[X]: \|u\| = 1\} = U$. Equation (2.16) then implies that

$$|h_k| \leqslant \sum_{\substack{i=0 \\ i \neq k}}^{n+1} \frac{1}{m_i} + 1 - \frac{1}{m_k}, \qquad k = 0, 1, \ldots, n+1; k \neq j. \tag{2.17}$$

Since it is known [1] that

$$\sum_{i=0}^{n+1} \frac{1}{m_i} = 1, \tag{2.18}$$

(2.17) and (2.18) imply that

$$|h_k| \leqslant 2\left(1 - \frac{1}{m_k}\right), \qquad k = 0, 1, \ldots, n+1; k \neq j. \tag{2.19}$$

Thus if $g \in U$, it follows easily from (2.15), (2.19), and the definition of m_k that for every j, $j = 0, 1, \ldots, n+1$, we have

$$\begin{aligned} \|B_n(g, X_n)\| &\leqslant \max_{k=0,1,\ldots,n+1} \frac{2\,|q_k(x_k)|}{1 + |q_k(x_k)|} \Lambda_{n+1}^j \\ &= \frac{2M_n^n(X_n)}{1 + M_n^n(X_n)} \Lambda_{n+1}^j. \end{aligned} \tag{2.20}$$

We conclude from (1.12) and (2.20) that

$$\lambda_n(X_n) \leqslant \frac{2M_n^n(X_n)}{1+M_n^n(X_n)} \min_{0 \leqslant j \leqslant n+1} A_{n+1}^j. \tag{2.21}$$

In order to establish the lower bound, consider $g \in C[X]$ such that $g_j = 0$. Then for each j, $j = 0, 1,..., n+1$, the system of equations (2.16) is given by

$$\begin{bmatrix} 1-\frac{1}{m_0} & \frac{1}{m_1} & -\frac{1}{m_2} & & \cdots & & \frac{(-1)^n}{m_{n+1}} \\ \frac{1}{m_0} & 1-\frac{1}{m_1} & \frac{1}{m_2} & & \cdots & & \frac{(-1)^{n+1}}{m_{n+1}} \\ & & & \cdots & & & \\ \frac{(-1)^j}{m_0} & \frac{(-1)^{j+1}}{m_1} & \cdots & 1-\frac{1}{m_{j-1}} & & \cdots & \frac{(-1)^{n+j+1}}{m_{n+1}} \\ \frac{(-1)^{j+2}}{m_0} & \frac{(-1)^{j+3}}{m_1} & & \cdots & 1-\frac{1}{m_{j+1}} & \cdots & \frac{(-1)^{n+j+3}}{m_{n+1}} \\ & & & \cdots & & & \\ \frac{(-1)^{n+2}}{m_0} & \frac{(-1)^{n+3}}{m_1} & & & \cdots & & 1-\frac{1}{m_{n+1}} \end{bmatrix} \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_{j-1} \\ g_{j+1} \\ \vdots \\ g_{n+1} \end{bmatrix} = \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{j-1} \\ h_{j+1} \\ \vdots \\ h_{n+1} \end{bmatrix} \tag{2.22}$$

Let A_j be the coefficient matrix in (2.22), $G_j = [g_0,..., g_{j-1}, g_{j+1},..., g_{n+1}]^{\mathrm{T}}$ and $H_j = [h_0,..., h_{j-1}, h_{j+1},..., h_{n+1}]^{\mathrm{T}}$.

Then (2.22) may be rewritten as

$$A_j G_j = H_j, \qquad j = 0, 1,..., n+1. \tag{2.23}$$

It follows from (2.18) that A_j is a strictly diagonally dominant matrix. Therefore we may conclude from (2.18) and [18, Theorem A] that

$$\|A_j^{-1}\|_\infty \leqslant m_j, \qquad j = 0, 1,..., n+1. \tag{2.24}$$

Now for fixed j, choose $\bar{x} \in X$ such that

$$A_{n+1}^j = \left\| \sum_{\substack{i=0 \\ i \neq j}}^{n+1} |l_i^j| \right\| = \sum_{\substack{i=0 \\ i \neq j}}^{n+1} |l_i^j(\bar{x})|, \tag{2.25}$$

and choose

$$h_i = \operatorname{sgn} l_i^j(\bar{x}), \qquad i = 0, 1,..., n+1;\ i \neq j. \tag{2.26}$$

For this choice of H_j we see from (2.15) that there exists $g \in C[X]$ such that (2.23) is satisfied, $g_j = 0$, and $g/\|G_j\|_\infty \in U$. Thus (2.15), (2.25), and (2.26) imply that

$$\|G_j\|_\infty \left\| B_n\left(\frac{g}{\|G_j\|_\infty}, X_n\right)\right\| = \Lambda_{n+1}^j. \tag{2.27}$$

It follows from (2.23), (2.24), and (2.27) that for each value of j, $j = 0, 1, \ldots, n+1$,

$$\|B_n(\cdot, X_n)\| \geqslant \frac{\Lambda_{n+1}^j}{m_j}. \tag{2.28}$$

The lower bounds for $\hat{\lambda}_n(X_n)$ follow from (1.12), (1.15), (2.18), and (2.28).

If $E_n(f) = X_n$ for $f \in C[I]$, then since $D_f B_n$ is a projection from $C[I]$ onto Π_n, $\|D_f B_n\| = \hat{\lambda}_n(X_n) \geqslant (2/\pi^2) \log n + O(1)$ [13]. We use Theorem 4 in the next example to illustrate that $\hat{\lambda}_n(X_n)$ may have an exponential rate of growth.

EXAMPLE 3. Let $X = I$, and suppose $X_n = \{x_i\}_{i=0}^{n+1}$, where $x_i = -1 + 2i/(n+1)$, $i = 0, 1, \ldots, n+1$. From (2.2) it follows that

$$\Lambda_{n+1}^k \geqslant \sum_{\substack{i=0\\ i\neq k}}^{n+1} \frac{|w_k(\bar{x})|}{|\bar{x} - x_i|\ |w_k'(x_i)|}, \qquad k = 0, 1, \ldots, n+1, \tag{2.29}$$

where $\bar{x} = (x_{n+1} + x_n)/2$. From (2.29) it can be shown that

$$\Lambda_{n+1}^k \geqslant \frac{1}{2^n} \sum_{\substack{i=0\\ i\neq k}}^{n+1} \frac{|k-i|}{|2n-2i+1|} \frac{\prod_{j=0, j\neq k}^{n+1} |2n-2j+1|}{\prod_{j=0, j\neq i}^{n+1} |i-j|}, \qquad k = 0, 1, \ldots, n+1.$$

Further simplification yields

$$\Lambda_{n+1}^k \geqslant \frac{(2n+1)!}{2^{2n} n!\ |2n-2k+1|} \sum_{\substack{i=0\\ i\neq k}}^{n+1} \frac{|k-i|}{|2n-2i+1|} \frac{1}{\prod_{j=0, j\neq i}^{n+1} |i-j|},$$

$$k = 0, 1, \ldots, n+1.$$

This inequality implies that

$$\Lambda_{n+1}^k \geqslant \frac{(2n)!}{2^{2n} n!\ (n+1)!\ |2n-2k+1|} \sum_{\substack{i=0\\ i\neq k}}^{n+1} \binom{n+1}{i}, \quad k = 0, 1, \ldots, n+1. \tag{2.30}$$

For n sufficiently large, (2.30) implies that

$$\begin{aligned}\Lambda_{n+1}^{k} &\geqslant \frac{(2n)!}{2^{2n+1}n!\,(n+1)!\,|2n-2k+1|}\sum_{i=0}^{n+1}\binom{n+1}{i}\\ &=\frac{(2n)!}{2^{n}n!\,(n+1)!\,|2n-2k+1|}.\end{aligned}$$

Applying the weak form of Stirling's formula [15, p. 98] yields

$$\Lambda_{n+1}^{k} \geqslant \frac{2^{n}}{en(n+1)\,|2n-2k+1|}, \qquad k=0, 1, \ldots, n+1. \tag{2.31}$$

Inequality (2.31) and Theorem 4 imply that

$$\hat{\lambda}_{n}(X_n) \geqslant \frac{2^{n}}{en(n+1)(n+2)(2n+1)}.$$

Although the authors have not found examples which demonstrate that the lower bounds given by (1.14) and (2.13) are sharp, equality (1.12) and the examples in this section do give a sense of the asymptotic behavior of the local Lipschitz constant.

Much less is known about global Lipschitz constants. The main theorem of the next section deals with the asymptotic behavior of the global Lipschitz constant.

3. Global Lipschitz Constants

The major difficulty in analyzing the global Lipschitz constant is the absence of any characterization of the global Lipschitz constant similar to (1.12) for the local Lipschitz constant or (1.13) for the strong unicity constant. In fact, whereas (1.12) and (1.13) imply that the local Lipschitz and strong unicity constants depend only on the set of extremal points X_n of $e_n(f)$, the global Lipschitz constant appears to depend on both f and the extremal points of $e_n(f)$.

The goal of the present section is to show that local and global Lipschitz constants may have very different behavior. This contrasts sharply with the fact that the local and global strong unicity constants are equal in the linear case [9].

Before exhibiting Theorem 5, we present some machinery that will allow us to relax the requirement that $|E_n(f)| = n+2$.

LEMMA 1 [17]. *For* $x \in X$, *let* $\sigma_f(x) = \operatorname{sgn}(f - B_n(f))(x)$. *Let* S *be the set of all ordered sequences* Y: $y_0 < y_1 < \cdots < y_n$ *of* $n+1$ *points in* $E_n(f)$ *such that* (a) $\sigma_f(y_0), \dots, \sigma_f(y_n)$ *alternate in sign, or* (b) *for some* $i = 1, \dots, n$, $\sigma_f(y_0), \dots, \sigma_f(y_{i-1})$ *alternate in sign,* $\sigma_f(y_i), \dots, \sigma_f(y_n)$ *alternate in sign, and* $\sigma_f(y_{i-1}) = \sigma_f(y_i)$. *For* Y: $y_0 < y_1 < \cdots < y_n$ *in* S, *let* p_Y *be the element of* Π_n *such that* $p_Y(y_i) = \sigma_f(y_i)$, $i = 0, \dots, n$. *Then*

$$M_n(f) = \max\{\|p_Y\| : Y \in S,\ \sigma_f(y)\, p_Y(y) \leqslant 1 \text{ for all } y \in E_n(f)\}. \tag{3.1}$$

We note that if $|E_n(f)| = n+2$, then (3.1) reduces to (1.13). If $|E_n(f)| > n+2$, Lemma 1 shows that the strong unicity constant is determined by $E_n(f)$ and the sign orientation of $e_n(f)$ on $E_n(f)$. Thus, any two functions possessing the same extremal set and sign orientation (in the sense of (a) or (b) in Lemma 1) generate the same strong unicity constant. These observations motivate the following definition of an extended global Lipschitz constant.

For $f \in C[I]$, let

$$\begin{aligned} G(f)(x) &= \sigma_f(x), && x \in E_n(f) \\ &= 0, && x \in I - E_n(f) \end{aligned} \tag{3.2}$$

and set

$$\mathbf{F}(f) = \{\bar{f} \in C[I] : G(\bar{f})(x) = G(f)(x) \text{ for all } x \in I \text{ or } G(\bar{f})(x) = -G(f)(x) \text{ for all } x \in I\}. \tag{3.3}$$

DEFINITION 6. For $f \in C[I]$, the extended global Lipschitz constant $\lambda_n^*(f)$ is defined by

$$\lambda_n^*(f) = \sup\{\lambda_n(h) : h \in \mathbf{F}(f)\}. \tag{3.4}$$

It follows from (1.7) and the remarks following Lemma 1 that

$$\lambda_n^*(f) \leqslant 2M_n(f). \tag{3.5}$$

The next theorem exhibits a lower bound for the extended global Lipschitz constant.

THEOREM 5. *For any* $f \in C[I]$,

$$\lambda_n^*(f) \geqslant M_n(f). \tag{3.6}$$

Proof. Let $f \in C[I]$. Without loss of generality, we may assume that

$n \geqslant 1$ and $B_n(f) \equiv 0$. Let $E_n(f) = \bigcup_{j=0}^{l} E_j$, where for $\gamma = \pm 1$, if $x \in E_j$, $0 \leqslant j \leqslant l$, then

$$f(x) = \gamma(-1)^j \, \|f\|, \tag{3.7}$$

and $x < y$ if $x \in E_m$ and $y \in E_{m+1}$. Note that $l \geqslant n+1$.

In this setting, Lemma 1 implies that there exists a $q_n \in \Pi_n$ and $y_0 < y_1 < \cdots < y_n$ such that $\{y_i\}_{i=0}^{n}$ satisfies one of the following two conditions:

(A) $f(y_i)$, $0 \leqslant i \leqslant n$, alternate in sign, or

(B) for some $k \geqslant 1$, $f(y_i)$ alternate in sign for $0 \leqslant i \leqslant k-1$, $f(y_i)$ alternate in sign for $k \leqslant i \leqslant n$, and $\operatorname{sgn} f(y_{k-1}) = \operatorname{sgn} f(y_k)$.

Furthermore, in either case (A) or case (B), $\|q_n\| = M_n(f)$,

$$q_n(x) \operatorname{sgn} f(x) \leqslant 1, \qquad x \in E_n(f), \tag{3.8}$$

and

$$q_n(y_i) = \operatorname{sgn} f(y_i), \qquad i = 0, \ldots, n. \tag{3.9}$$

We note for $n \geqslant 1$ Lemma 1 implies that q_n is not constant.

Since cases (A) and (B) are similar, we shall consider only case (B). To establish (3.6) we construct two functions, $\bar{f}$ and f_1. The function $\bar{f}$ will be shown to be an element of $\mathbf{F}(f)$, and f_1 will be used to estimate the size of $\lambda_n(\bar{f})$.

First set $x_i = y_i$, $0 \leqslant i \leqslant k-1$, and $x_i = y_{i-1}$, $k+1 \leqslant i \leqslant n+1$, where $\{y_i\}_{i=0}^{n}$ is described in (B). The point x_k will be defined momentarily.

If $0 \leqslant i \leqslant n+1$ and $i \neq k$, we have from (B) that

$$\operatorname{sgn} f(x_i) = \mu(-1)^i, \tag{3.10}$$

where $\mu = \pm 1$. Therefore from (3.9),

$$q_n(x_i) = (-1)^i, \qquad 0 \leqslant i \leqslant n+1, \; i \neq k. \tag{3.11}$$

Assume that $x_{k-1} \in E_r$ and $x_{k+1} \in E_s$, where $r < s$. We claim that x_{k-1} is the largest point in E_r and x_{k+1} is the smallest point in E_s. By way of contradiction, assume there exists an $\bar{x} \in E_r$ with $\bar{x} > x_{k-1}$. (The argument is similar if we assume there exists a $\bar{y} \in E_s$ with $\bar{y} < x_{k+1}$.) From (3.7), (3.8), and (3.10) we have that

$$\mu(-1)^{k-1} \, q_n(\bar{x}) \leqslant 1. \tag{3.12}$$

By (3.11),

$$q_n(x_{k-1}) = q_n(x_{k+1}) = \mu(-1)^{k-1}. \tag{3.13}$$

On $[x_0, x_{k-1}]$, (3.11) implies that q_n has $k-1$ zeroes $t_1 < t_2 < \cdots < t_{k-1}$. Thus q_n' has $k-2$ distinct zeroes in the open interval (t_1, t_{k-1}). Analogously, q_n has $n-k$ zeroes $s_1 < \cdots < s_{n-k}$ in $[x_{k+1}, x_{n+1}]$, so q_n' has $n-k-1$ zeroes in the open interval (s_1, s_{n-k}).

Moreover, since (3.12) and (3.13) hold, q_n' must have at least three zeroes in the open interval (t_{k-1}, s_1). Thus q_n' has at least n zeroes in (x_0, x_{n+1}), which contradicts the assumption that q_n is not constant. This establishes our claim.

Now let $x_k = x_{k-1} + \delta$, where $\delta > 0$ is sufficiently small that $x_k < x_{k+1}$; thus $x_k \notin E_n(f)$. Using an argument similar to the above with $\bar{x} = x_k$, it can be shown that

$$|q_n(x_k)| \geqslant 1. \tag{3.14}$$

Using the fact that $M_n(\alpha f) = M_n(f)$ if $\alpha \neq 0$, we may also assume that

$$\|f\| \geqslant 2M_n(f). \tag{3.15}$$

From the definition of q_n and (3.15), it follows that

$$\|f\| \geqslant 2\,\|q_n\| \geqslant 2. \tag{3.16}$$

We are now ready to define the functions $\bar{f}$ and f_1 on $E_n^* = E_n(f) \cup \{x_k\}$. We will eventually show $\bar{f} \in \mathrm{F}(f)$ and $\lambda_n(\bar{f}) \geqslant M_n(f) + O(\delta)$ for $\delta > 0$ sufficiently small, where $\bar{f}$ and f_1 are in $C[I]$. Let

$$\begin{aligned} \bar{f}(x) &= f(x), && x \in E_n(f) \\ &= (-1)^k(\|f\| - \delta), && x = x_k, \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} f_1(x) &= f(x) - \operatorname{sgn} f(x), && x \in E_n(f) \\ &= (-1)^k\,\|f\| - q_n(x_k), && x = x_k. \end{aligned} \tag{3.18}$$

For $x \in E_n^*$, (3.13), (3.14), (3.17), and (3.18) imply that

$$|f_1(x) - \bar{f}(x)| \leqslant \delta + |q_n(x_k)| = 1 + O(\delta). \tag{3.19}$$

Furthermore, from (3.10), (3.11), and (3.18) we have that

$$f_1(x_i) + q_n(x_i) = \mu(-1)^i\,\|f\|, \qquad 0 \leqslant i \leqslant n+1. \tag{3.20}$$

If $x \in E_n(f)$, then (3.8) and (3.16) insure that $-\|f\| \leqslant q_n(x) \operatorname{sgn} f(x) \leqslant 1$;

also, (3.18) implies that $|f_1(x)+q_n(x)| = |\,\|f\|-1+q_n(x)\operatorname{sgn} f(x)|$. Together with (3.20), this shows that for all $x \in E_n^*$,

$$|f_1(x)+q_n(x)| \leqslant \|f\|. \tag{3.21}$$

Inequalities (3.20) and (3.21) now imply that $B_n(f_1, E_n^*) = -q_n$.

We have constructed functions $\bar{f}$ and f_1 on E_n^* with best approximations $B_n(\bar{f}, E_n^*) \equiv 0$ (from 3.17) and $B_n(f_1, E_n^*) = -q_n$. To achieve the objectives described below (3.9), we must extend both $\bar{f}$ and f_1 to all of I while preserving their best approximations. We must also insure that $\bar{f} \in \mathbf{F}(f)$, and preserve (3.19) for all $x \in I$. Since the complete extensions can be obtained in a standard (but somewhat technical) way, we only sketch the extension process.

In this context, let (α, β) be any open interval such that $\alpha, \beta \in E_n^* \cup \{-1, 1\}$ and $(\alpha, \beta) \cap E_n^* = \varnothing$. If $\alpha, \beta \in E_n^*$, set $\bar{f}$ and f_1 equal to zero on $[\alpha+\varepsilon, \beta-\varepsilon]$ for some ε with $0<\varepsilon<(\beta-\alpha)/2$, and let $\bar{f}$ and f_1 be linear on $[\alpha, \alpha+\varepsilon]$ and on $[\beta-\varepsilon, \beta]$. If $\alpha=-1 \notin E_n^*$, replace $\alpha+\varepsilon$ above by -1, and if $\beta=1 \notin E_n^*$ then replace $\beta-\varepsilon$ above by 1. Extending $\bar{f}$ to all of I in this manner insures that $B_n(\bar{f}, I) \equiv 0$ and $G(\bar{f})(x) = G(f)(x)$ for all $x \in I$. We proceed to show that $B_n(f_1, I) = -q_n$ and that (3.19) holds for all $x \in I$.

First assume that $\alpha \in E_n(f)$, $\beta \in E_n^*$, and $f(\alpha) = \|f\|$ (the case $\alpha \in E_n(f)$, $\beta \in E_n^*$, and $f(\alpha) = -\|f\|$ is similar). Then (3.8) and (3.16) imply that $-\|f\| \leqslant q_n(\alpha) \leqslant 1$. If $q_n(\alpha)<1$, then using (3.16) and (3.18) we have $-\|f\| < f_1(\alpha)+q_n(\alpha) = \|f\|-1+q_n(\alpha) < \|f\|$. Thus for $\varepsilon>0$ sufficiently small, (3.21) holds on $(\alpha, \alpha+\varepsilon)$, and thus on $(\alpha, \beta-\varepsilon)$.

If $q_n(\alpha)=1$, choose $\varepsilon>0$ small enough to insure that $q_n(x)>0$ for all $x \in (\alpha, \alpha+\varepsilon)$. If there exists an $x \in (\alpha, \alpha+\varepsilon)$ such that (3.21) fails, then $\|f\| < |f_1(x)+q_n(x)| = f_1(x)+q_n(x)$. Hence

$$\begin{aligned} q_n(\alpha)-q_n(x) &= 1-q_n(x) < 1+f_1(x)-\|f\| \\ &= \frac{\|f\|-1}{\varepsilon}(\alpha+\varepsilon-x)-(\|f\|-1) = \frac{\|f\|-1}{\varepsilon}(\alpha-x). \end{aligned}$$

Therefore

$$\frac{q_n(x)-q_n(\alpha)}{x-\alpha} > \frac{\|f\|-1}{\varepsilon},$$

which is false if ε is sufficiently small, since for some $\xi \in (\alpha, \alpha+\varepsilon)$,

$$\left|\frac{q_n(x)-q_n(\alpha)}{x-\alpha}\right| = |q_n'(\xi)| \leqslant \|q_n'\|.$$

Next assume that $\alpha = x_k$, $\beta \in E_n(f)$, and

$$f_1(\alpha) = \|f\| - q_n(\alpha) \tag{3.22}$$

(the case $f_1(\alpha) = -\|f\| - q_n(\alpha)$ is similar). Equalities (3.10), (3.18), and (3.22) imply that $\operatorname{sgn} f(x_{k-1}) = -1$, so (3.9) yields $q_n(x_{k-1}) = -1$. Since $q_n \in C[I]$, $q_n(x_k) = -1 + O(\delta)$. For ε and δ sufficiently small we can thus assume that $f_1(x) > 0$ and that $-\|q_n\| \leqslant q_n(x) < 0$ on $(\alpha, \alpha + \varepsilon)$. If (3.21) fails for some $x \in (\alpha, \alpha + \varepsilon)$, then $\|f\| < |f_1(x) + q_n(x)|$. Assuming $f_1(x) < -q_n(x)$ implies $\|f\| < -f_1(x) - q_n(x) \leqslant \|q_n\|$, contradicting (3.16). On the other hand, if $f_1(x) \geqslant -q_n(x)$, then

$$\begin{aligned}\|f\| < f_1(x) + q_n(x) &= \frac{f_1(\alpha)}{\varepsilon}(\alpha + \varepsilon - x) + q_n(x)\\ &= f_1(\alpha) - \frac{f_1(\alpha)}{\varepsilon}(x - \alpha) + q_n(x).\end{aligned}$$

Utilizing (3.22) now yields

$$\|f\| < \|f\| - q_n(\alpha) - \frac{f_1(\alpha)}{\varepsilon}(x - \alpha) + q_n(x),$$

which in turn implies that

$$\frac{q_n(x) - q_n(\alpha)}{x - \alpha} > \frac{f_1(\alpha)}{\varepsilon},$$

which is again a contradiction for ε sufficiently small. Therefore (3.21) is again valid for the interval $(\alpha, \beta - \varepsilon)$. Similar arguments establish (3.21) for the interval $[\beta - \varepsilon, \beta]$, as well as for the cases $\alpha = -1 \notin E_n^*$ and $\beta = 1 \notin E_n^*$. We have extended f_1 to all of I so that $B_n(f_1, I) = -q_n$, and so that (3.19) is true for any $x \in I$. Thus

$$\begin{aligned}\lambda_n(\bar{f}) &\geqslant \frac{\|B_n(\bar{f}, I) - B_n(f_1, I)\|}{\|\bar{f} - f_1\|} = \frac{\|q_n\|}{\|\bar{f} - f_1\|}\\ &\geqslant \frac{M_n(f)}{1 + O(\delta)} = M_n(f) + O(\delta)\end{aligned} \tag{3.23}$$

for δ sufficiently small. Since $\bar{f} \in \mathrm{F}(f)$, (3.23) implies that $\lambda_n^*(f) \geqslant M_n(f)$, which concludes the proof of Theorem 5. ∎

COROLLARY 1. *For $f \in C[I]$,*

$$M_n(f) \leqslant \lambda_n^*(f) \leqslant 2M_n(f). \tag{3.24}$$

The inequalities in (3.24) show that the extended global Lipschitz constant is asymptotically equivalent to the strong unicity constant.

COROLLARY 2. *For any* $f \in C[I]$ *with extremal set* $E_n(f)$, *there exists* $g_\delta \in C[I]$ *such that* $E_n(g_\delta) = E_n(f)$, $e_n(f) \cdot e_n(g_\delta) > 0$ *on* $E_n(f)$, $M_n(g_\delta) = M_n(f)$, *and* $\lambda_n(g_\delta) \geqslant M_n(g_\delta) + O(\delta)$ *for* $\delta > 0$ *sufficiently small.*

4. Observations and Conclusions

Example 1 exhibits a function $f \in C[I]$ such that $|E_n(f)| = |X_n| = n+2$ for all $n \geqslant 1$ and

$$\lim_{n \to \infty} \frac{\hat{\lambda}_n(X_n)}{M_n(X_n)} = 0. \tag{4.1}$$

If $|X_n| = n+2$, then (3.3) and (3.4) imply that X_n determines an extended global Lipschitz constant defined by

$$\lambda_n^*(X_n) = \sup\{\lambda_n(h)\colon h \in C[X] \text{ and } E_n(h) = X_n\}.$$

Thus for the point sets $X_n = E_n(f)$ in Example 1, (3.24) and (4.1) show that

$$\lim_{n \to \infty} \frac{\hat{\lambda}_n(X_n)}{\lambda_n^*(X_n)} = 0. \tag{4.2}$$

From Example 1, the quotient in (4.1) converges to zero like $n^{-1} \log(n+1)$. Our last example will demonstrate that the quotient $\hat{\lambda}_n(X_n)/M_n(X_n)$ may converge to zero arbitrarily rapidly.

EXAMPLE 4. Let $X_n \subseteq X$ consist of the points $x_0 < x_1 < \cdots < x_n < x_{n+1}$ and let q_j, $j = 0, 1, \ldots, n+1$, be defined by (1.10). Then for $0 \leqslant j \leqslant n$,

$$|q_j(x)| \leqslant \sum_{\substack{i=0 \\ i \neq j}}^{n+1} |w_j(x)/[(x - x_i)\, w_j'(x_i)]|,$$

where w_j is defined below (2.2). Therefore

$$\begin{aligned} |q_j(x)| &\leqslant 2^n \sum_{\substack{i=0 \\ i \neq j}}^{n+1} \frac{1}{|w_j'(x_i)|} \\ &\leqslant 2^n (x_n - x_0) \sum_{i=0}^{n} \frac{1}{|w'(x_i)|} + \frac{2^n}{(x_{n+1} - x_n)^n}. \end{aligned} \tag{4.3}$$

For fixed $\varepsilon > 0$ we now require that

$$\varepsilon < x_{n+1} - x_n < 1 \tag{4.4}$$

and

$$x_n - x_0 \leqslant \delta < 1, \tag{4.5}$$

where $\delta > 0$ will be specified later. With these additional assumptions, (4.3) implies that

$$|q_j(x)| \leqslant \frac{2^{n+1}(x_n - x_0)}{(x_{n+1} - x_n)^n} \sum_{i=0}^{n} \frac{1}{|w'(x_i)|}. \tag{4.6}$$

For $j = n+1$,

$$|q_{n+1}(x)| \leqslant \sum_{i=0}^{n} \left| \frac{w_{n+1}(x)}{(x - x_i)\, w'_{n+1}(x_i)} \right| \leqslant 2^{n+1} \sum_{i=0}^{n} \frac{1}{|w'(x_i)|}, \tag{4.7}$$

while

$$\begin{aligned} |q_{n+1}(x_{n+1})| &= \left| \sum_{i=0}^{n} \frac{(-1)^i\, w_{n+1}(x_{n+1})}{(x_{n+1} - x_i)\, w'_{n+1}(x_i)} \right| \\ &\geqslant (x_{n+1} - x_n)^{n+1} \sum_{i=0}^{n} \frac{1}{|w'(x_i)|}. \end{aligned} \tag{4.8}$$

From (1.12) we see that

$$\hat{\lambda}_n(X_n) \leqslant \sum_{j=0}^{n} \|q_j\| + \frac{\|q_{n+1}\|}{|q_{n+1}(x_{n+1})|}. \tag{4.9}$$

Therefore the inequalities (4.6) through (4.9) imply that

$$\hat{\lambda}_n(X_n) \leqslant \frac{(n+1)\, 2^{n+1}(x_n - x_0)}{(x_{n+1} - x_n)^n} \sum_{i=0}^{n} \frac{1}{|w'(x_i)|} + \frac{2^{n+1}}{(x_{n+1} - x_n)^{n+1}}.$$

Applying (4.4) and (4.5) to this inequality yields

$$\hat{\lambda}_n(X_n) \leqslant \frac{2^{n+1}(n+2)(x_n - x_0)}{(x_{n+1} - x_n)^{n+1}} \sum_{i=0}^{n} \frac{1}{|w'(x_i)|}. \tag{4.10}$$

From (1.13) and (4.8) we see that

$$M_n(X_n) \geqslant |q_{n+1}(x_{n+1})| \geqslant (x_{n+1} - x_n)^{n+1} \sum_{i=0}^{n} \frac{1}{|w'(x_i)|}. \tag{4.11}$$

Thus (4.10) and (4.11) yield

$$\frac{\hat{\lambda}_n(X_n)}{M_n(X_n)} \leqslant \frac{(x_n - x_0) \cdot 2^{n+1} \cdot (n+2)}{(x_{n+1} - x_n)^{2n+2}}. \tag{4.12}$$

If we let $\delta = \mu^{2n+2}/(n+2)\,2^{n+1}$, where μ can be chosen to be arbitrarily small, then (4.4), (4.5), and (4.12) imply that

$$\frac{\hat{\lambda}_n(X_n)}{M_n(X_n)} \leqslant \left(\frac{\mu}{\varepsilon}\right)^{2n+2}.$$

Thus for an appropriate choice of X_n, $n = 1, 2, \ldots$, the quotients $\hat{\lambda}_n(X_n)/M_n(X_n)$ will converge to zero arbitrarily rapidly.

Inequality (3.24) and Example 4 demonstrate that the quotient $\hat{\lambda}_n(X_n)/\lambda_n^*(X_n)$ also may converge to zero arbitrarily rapidly. Thus, in contrast to the equality of the local and global strong unicity constants in the setting of this paper, the local and extended global Lipschitz constants may have very different asymptotic behavior.

Although Corollaries 1 and 2 do shed considerable light on the behavior of the global Lipschitz constant, they leave unanswered the question of whether or not $M_n(f)$ and $\lambda_n(f)$ have the same asymptotic order for every $f \in C[X]$. In this regard, it appears that the global Lipschitz constant $\lambda_n(f)$ really does depend on the function f as well as on the sign orientation of the error function on its extremal set. On the other hand, Theorem 5 and (3.5) establish that $\lambda_n^*(f)$ and $M_n(f)$ always have the same asymptotic behavior.

REFERENCES

1. J. R. ANGELOS, M. S. HENRY, E. H. KAUFMAN, JR., AND T. D. LENKER, Local Lipschitz constants, *J. Approx. Theory* **43** (1985), 53–63.
2. M. W. BARTELT, On Lipschitz conditions, strong unicity and a theorem of A. K. Cline, *J. Approx. Theory* **14** (1975), 245–250.
3. M. W. BARTELT AND M. S. HENRY, Continuity of the strong unicity constant on $C[X]$ for changing X, *J. Approx. Theory* **28** (1980), 87–95.
4. E. W. CHENEY, "Introduction to Approximation Theory," McGraw–Hill, New York, 1966.
5. E. W. CHENEY AND T. J. RIVLIN, Some polynomial approximation operators, *Math. Z.* **145** (1975), 33–42.
6. G. FREUD, Eine ungleichung fur Tschebyscheffsche approximations polynome, *Acta Sci. Math. (Szeged)* **19** (1958), 162–164.
7. M. S. HENRY, E. H. KAUFMAN, JR., AND T. D. LENKER, Lipschitz constants for small perturbations, *in* "Approximation Theory IV" (C. Chui, L. Schumaker, and J. Ward, Eds.), pp. 515–520, Academic Press, New York, 1983.

8. M. S. Henry and J. A. Roulier, Lipschitz and strong unicity constants for changing dimension, *J. Approx. Theory* **22** (1978), 85–94.
9. M. S. Henry, D. P. Schmidt, and J. J. Swetits, Uniform strong unicity for rational approximation, *J. Approx. Theory* **33** (1981), 131–146.
10. M. S. Henry and J. J. Swetits, Limits of strong unicity constants for certain C^∞ functions, *Acta Math. Acad. Sci. Hungar.* **43** (1984), 309–323.
11. M. S. Henry and J. J. Swetits, Lebesgue constants for certain classes of nodes, *J. Approx. Theory* **39** (1983), 211–227.
12. M. S. Henry, J. J. Swetits, and S. E. Weinstein, Orders of strong unicity constants, *J. Approx. Theory* **31** (1981), 175–187.
13. A. Kroó, The Lipschitz constant of the operator of best approximation, *Acta Math. Acad. Sci. Hungar.* **35** (1980), 279–292.
14. A. Kroó, Differential properties of the operator of best approximation, *Acta Math. Acad. Sci. Hungar.* **30** (1977), 319–331.
15. T. J. Rivlin, "An Introduction to the Approximation of Functions," Blaisdell, Waltham, Mass., 1969.
16. D. P. Schmidt, On an unboundedness conjecture for strong unicity constants, *J. Approx. Theory* **24** (1978), 216–223.
17. D. P. Schmidt, A characterization of strong unicity constants, *in* "Approximation III" (E. W. Cheney, Ed.), pp. 805–810, Academic Press, New York, 1980.
18. R. S. Varga, On diagonal dominance arguments for bounding $\|A^{-1}\|_\infty$, *Linear Algebra Appl.* **14** (1976), 211–217.